ON THE THEORY OF COMPRESSIBLE, IDEALLY PLASTIC MEDIA

(K TEORII SZHIMAEMYKH IDEAL'NO PLASTICHESKIKH SRED)

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The plastic deformation of continuous media can be accompanied by a change of volume. The associated flow law [1] determines an irreversible change of volume which depends on the shape of the yield surface.

The "associated" compressibility of a material is a consequence of the shearing deformations and is in no way determined by the change of volume due to hydrostatic pressure.

Soils and related physical media undergo an irreversible change of volume when subjected to triaxial compression; this was studied in detail, for example, in [2]. In the note [3], consideration was given to a modification of a theorem of von Mises according to which it was possible to determine the relation between the first invariants of the tensors of stress and strain independently of the form of the yield surface. However, the equations appearing in the stress-strain law proposed in [3] have a serious drawback: the characteristic surfaces of the equations determining the states of stress and strain turn out to be different in the general case. Consequently, the boundary conditions, which are given on prescribed parts of the surface of the body, determine different regions of existence of the solutions for the stresses and the displacement velocities. These regions, according to [3], coincide only for materials whose yield condition does not depend on the first invariant of the stress tensor.

Below, we will derive general relations between the states of stress and strain for arbitrary isotropic, ideally plastic media, leading to the previously given dependence of the plastic volumetric deformation on the hydrostatic pressure and for which the characteristic surfaces of the equations determining the states of stress and strain are identical.

1. We will assume that the yield condition of the material has been given in the form

$$\Phi_i(\sigma, \Sigma_2, \Sigma_3) = 0 \qquad (i = 1, \ldots, n) \tag{1.1}$$

where σ is the first invariant of the stress tensor, and Σ_2, Σ_3 are the second and third invariants, respectively, of the stress deviator.

According to the associated flow law of von Mises, we have

$$de_{ij} = d\lambda_k \frac{\partial \Phi_k}{\partial \sigma_{ij}} = d\lambda_k \left[\frac{1}{3}\frac{\partial \Phi_k}{\partial \sigma} + \frac{\partial \Phi_k}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi_k}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right] \tag{1.2}$$

where e_{ij} are the components of the strain tensor, and in (1.2) we have summation over the index k. From condition (1.2) it can be found that

$$de_{ij}' = d\lambda_k \left[\frac{\partial \Phi_k}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi_k}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right] \tag{1.3}$$

$$de = \frac{1}{3} d\lambda_k \frac{\partial \Phi_k}{\partial \sigma}, \qquad e = \frac{1}{3}(e_{11} + e_{22} + e_{33}) \tag{1.4}$$

Here and henceforth, the components of a deviator are denoted by a prime. It is to be observed that

$$d\lambda_k = 0 \text{ when } \Phi_k < 0 \tag{1.5}$$

According to [3], if a dependence $e = \varphi(\sigma)$ has been given, then the equations of the law connecting $e_{ij} - \sigma_{ij}$ can, by condition (1.1), be written down in the form

$$de_{ij}' = d\lambda_k \left[\frac{\partial \Phi_k}{\partial \Sigma_2}\frac{\partial \Sigma_2}{\partial \sigma_{ij}} + \frac{\partial \Phi_k}{\partial \Sigma_3}\frac{\partial \Sigma_3}{\partial \sigma_{ij}}\right], \qquad e = \varphi(\sigma) \qquad \left(de = \frac{d\varphi}{d\sigma} d\sigma\right) \tag{1.6}$$

Thus, the difference in the stress-strain relation in both cases consists in the determination of the law of compressibility (1.4) and (1.6).

In the following it will be convenient to make use of the expressions for the components of the strain rate

$$\varepsilon_{ij} = \frac{de_{ij}}{dt}, \qquad \mu_k = \frac{d\lambda_k}{dt}$$

It is well-known that the characteristic surfaces of equations, determining the states of stress and strain according to (1.3) and (1.4), are identical.

However, in the case of equations (1.3) and (1.6) the characteristic surfaces of the equations determining the state of strain agree with those of the equations determining the state of stress only if the functions Φ_k do not depend on σ.

We will show this with the example of states of stress and strain corresponding, in the space of principal stresses σ_1, σ_2, σ_3, to the edge of a curvilinear pyramid, the equation of which is given in the form

$$\Phi \equiv \max |\Psi(\tau_\nu, \sigma_\nu)| = 0 \tag{1.7}$$

where τ_ν, σ_ν are the shear and normal stress, respectively, acting on the area with normal ν.

We will consider some edge whose equation will be written in the form [4]

$$\sigma_1 = \sigma_2, \qquad \sigma_3 = f(\sigma_1) \tag{1.8}$$

Passing to the components in a Cartesian system of coordinates, relations (1.8) will be written down in the form

$$\begin{gathered}[\sigma_x - g(\sigma)][\sigma_y - g(\sigma)] - \tau_{xy}^2 = 0\\ [\sigma_y - g(\sigma)][\sigma_z - g(\sigma)] - \tau_{yz}^2 = 0\\ [\sigma_z - g(\sigma)][\sigma_x - g(\sigma)] - \tau_{zx}^2 = 0\end{gathered} \tag{1.9}$$

where

$$\sigma_1 = g(\sigma), \qquad 2\sigma_1 + f(\sigma_1) = 3\sigma$$

If we use relations (1.9) as the plastic potential, we will find

$$\begin{gathered}\varepsilon_x' = {}^1/_3\mu_1(2\sigma_y - \sigma_x - g) - {}^1/_3\mu_2(\sigma_y + \sigma_z - 2g) + {}^1/_3\mu_3(2\sigma_z - \sigma_x - g)\\ \varepsilon_y' = {}^1/_3\mu_1(2\sigma_x - \sigma_y - g) + {}^1/_3\mu_2(2\sigma_z - \sigma_y - g) - {}^1/_3\mu_3(\sigma_z + \sigma_x - 2g)\\ \varepsilon_z' = -{}^1/_3\mu_1(\sigma_x + \sigma_y - 2g) + {}^1/_3\mu_2(2\sigma_y - \sigma_z - g) + {}^1/_3\mu_3(2\sigma_x - \sigma_z - g)\\ \varepsilon_{xy} = -\mu_1\tau_{xy}, \qquad \varepsilon_{yz} = -\mu_2\tau_{yz}, \qquad \varepsilon_{zx} = -\mu_3\tau_{zx}\\ \varepsilon_x + \varepsilon_y + \varepsilon_z = (1 - dg/d\sigma)[\mu_1(\sigma_x + \sigma_y - 2g) + \mu_2(\sigma_y + \sigma_z - 2g) +\\ + \mu_3(\sigma_z + \sigma_x - 2g)]\end{gathered} \tag{1.10}$$

Eliminating the undetermined multipliers μ_i and changing over to the components of the displacement velocity u, v, w, equations (1.10) can be written in the form

$$\begin{gathered}2\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{n_2}{n_1} - \frac{n_1}{n_2}\right) + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{n_3}{n_2} - \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{n_3}{n_1} = 0\\ 2\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{n_1}{n_2} - \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\left(\frac{n_3}{n_2} - \frac{n_2}{n_3}\right) + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{n_1}{n_3} = 0\end{gathered} \tag{1.11}$$

$$\begin{gathered}2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \left(1 - \frac{dg}{d\sigma}\right)\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right) + \left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)\left(\frac{n_2}{n_3} + \frac{n_3}{n_2}\right) +\right.\\ \left. + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\left(\frac{n_1}{n_3} + \frac{n_3}{n_1}\right)\right] = 0\end{gathered} \tag{1.12}$$

The characteristic surfaces $\chi(x, y, z)$ of the system of equations (1.11), (1.12) satisfy the equation

$$F\left[\left(1+\frac{\partial f}{\partial \sigma_1}\right)F^2-(\text{grad }\chi)^2\right]=0, \qquad F=\frac{\partial \chi_i}{\partial x}n_i \tag{1.13}$$

Here, n_i are the direction-cosines of the principal stress σ_3 in the space of principal stresses.

Obviously, for the case of a stress-strain relation determined by equations (1.6), equations (1.11) remain valid but in place of condition (1.12) we will have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{d\varphi}{d\sigma}\frac{d\sigma}{dt} \tag{1.14}$$

The characteristic surfaces $\chi(x, y, z)$ of the system of equations (1.11) and (1.14) satisfy the equation

$$F\,[2F^2-(\text{grad }\chi)^2]=0 \tag{1.15}$$

2. It is essential to note that the compressibilities determined by expressions (1.4) and (1.6) are of completely different origin. The compressibility determined by equation (1.4) is connected essentially with the change of shape of the material. As a matter of fact, in relations (1.3) and (1.4) the quantity $d\lambda_k \neq 0$ only in the case, if $\Phi_k = 0$, $d\Phi_k/dt = 0$. Consequently, if the material is subjected to a hydrostatic pressure alone, then $\Phi_k < 0$, $d\Phi_k = 0$ always and, as a consequence of (1.3) and (1.4), the material will not be able to acquire any residual change of volume.

The compressibility determined by equations (1.6), on the contrary, is not at all connected with the change of shape and can be determined from experiments on uniform, triaxial compression.

From the consideration of equations (1.12) and (1.14), it follows that equation (1.4) has basic influence on the form of the characteristic surfaces $\chi(x, y, z)$. On the other hand, the form of the function $\varphi(\sigma)$ in (1.6) has no influence whatsoever on the form of the characteristic surfaces and only affects the form of relations along the characteristics. Relations (1.4) and (1.6) have invariant character. If the compressibility of the material is determined in the form

$$de=d\lambda_k\frac{\partial \Phi_k}{\partial \sigma}+\frac{d\varphi}{d\sigma}d\sigma \tag{2.1}$$

and this relation is combined with equation (1.3), the resulting relation firstly will determine the dependence $e = \varphi(\sigma)$ when there is a uniform state of triaxial pressure, and secondly the characteristic surfaces of the equations determining the states of stress and strain will be identical. For the above-considered case of the edge of a curvilinear pyramid it is, by virtue of (2.1), necessary to add to conditions (1.11) the condition

$$\varepsilon = \frac{1}{3}\left(1 - \frac{dg}{d\sigma}\right)\left[\varepsilon_{xy}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right) + \varepsilon_{yz}\left(\frac{n_2}{n_3} + \frac{n_3}{n_2}\right) + \varepsilon_{zx}\left(\frac{n_1}{n_3} + \frac{n_3}{n_1}\right)\right] + \frac{1}{3}\frac{d\varphi}{d\sigma}\frac{d\sigma}{dt} = 0 \tag{2.2}$$

It is easily seen that the additional term $(d\varphi/d\sigma)d\sigma$ affects only the relations along the characteristics.

We shall now carry out some transformations. For elastic-plastic media, one assumes

$$de_{ij} = de_{ij}^e + de_{ij}^p \tag{2.3}$$

where the indices e and p denote the components of the elastic and plastic strain, respectively.

The components of the elastic strain satisfy Hooke's law. The change of volume will be composed of the two parts

$$de = de^e + de^p \tag{2.4}$$

We note further that when the elastic strains in the body are neglected, there is no essential loss of generality in setting $d\sigma/dt = 0$. Therefore, the presence of the compressibility (1.6) in relation (2.1) does not make itself evident in the size of the limit loads.

It is to be observed that the expression $e_1 + e_2 + e_3$ is the scalar product of the displacement vector $e = e_1\mathbf{i} + e_2\mathbf{j} + e_3\mathbf{k}$ and the vector $r = \mathbf{i} + \mathbf{j} + \mathbf{k}$ directed along the line equally inclined to the axes of the principal strains e_1, e_2, e_3. Here, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors directed along the axes of e_1, e_2, e_3. Similarly, for the increments and rates of strain.

From relations (2.1) it follows that: for ideally plastic media, the material of which is capable of acquiring an irreversible change of volume under uniform triaxial compression, the velocity vector of the plastic deformation is not normal to the yield surface.

As is known, the postulate of Drucker [5,6] leads to the fact that the vector of the plastic strain rate lies along the gradient to the yield surface. However, Drucker's inference is based essentially on the assumption that in the zone bounded by the yield surface only elastic strains take place. In the present case it can be assumed that irreversible plastic strains can occur irrespective of whether the state of stress corresponds to the yield surface or not.

If Drucker's postulate is formulated only in regard to the components of the deviator of the strain rate, and one proceeds from the increment of work $\delta W = \sigma_{ij}\delta e_{ij}'$, it can be found as a consequence that the components of the strain-rate deviator are proportional to the partial derivatives with respect to the components of stress of the yield condition, which depends on the second and third invariants of the stress deviator (the first invariant σ in this case enters in the yield condition as a parameter). This fact is expressed by equations (1.3).

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